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Two loop $\overline{\text{MS}}$ Gribov mass gap equation with massive quarks

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Abstract. We compute the two loop $\overline{\text{MS}}$ correction to the Gribov mass gap equation in the Landau gauge using the Gribov-Zwanziger Lagrangian with massive quarks included. The computation involves dilogarithms of complex arguments and reproduces the known gap equation when the quark mass tends to zero.

1 Introduction.

The quantum field theory underlying the strong nuclear force is Quantum Chromodynamics (QCD). It is an extension of Quantum Electrodynamics (QED) where the gauge fields are required to be elements of a non-abelian colour group, $SU(3)$, as opposed to the abelian $U(1)$ of electric charge. Whilst this is a simple mathematical generalization the properties of the Yang-Mills field theory are significantly different. Clearly QCD is asymptotically free which is not unrelated to the fact that the basic fields analogous to electrons correspond to particles which are never isolated in nature, called quarks. They are held together in pairs or triplets by the quanta of the strong force called gluons. Equally these have never been seen isolated in experiments but rather at high energy they are effectively massless asymptotically free fields which to all intents and purposes behave as massless fundamental particles. To a degree this behaviour is parallel to the properties of the photons and electrons of QED. However, both fundamental forces differ in behaviour in the infrared region. For instance, in QCD infrared slavery dominates the confinement picture and the gluon propagator does not have the behaviour of a massless fundamental particle. One situation where this property can be manifestly seen is in Gribov's construction of the gluon propagator at low energy in the Landau gauge, [1]. An additional divergence in the structure of QED and QCD emanates from the way one tries to fix a (linear) covariant gauge. In QED one can fix the gauge in a global sense. By contrast, Gribov pointed out, [1], that in Yang-Mills theory the covariant gauge condition for the Landau gauge has an ambiguity. This occurs at zeroes of the Faddeev-Popov operator when different gauge configurations satisfy the *same* gauge fixing condition. In a local region in the neighbourhood of the origin of configuration space, where perturbation theory is valid, there is no such ambiguity and standard perturbative calculations are perfectly adequate to describe ultraviolet behaviour. However, to properly fix the gauge globally the problem of Gribov copies must be taken into account in defining the path integral of the theory, [1]. Gribov achieved this by restricting the path integral to the region of configuration space containing the first Gribov region, denoted by Ω. This is defined to be the region containing the origin where the Faddeev-Popov operator, $\mathcal{M}(A) \equiv - \partial^\mu D_\mu(A)$, is strictly positive. Consequently, the path integral is cutoff and a natural mass parameter, γ, called the Gribov mass emerges, [1]. It is not an independent parameter of Yang-Mills but is non-perturbative and satisfies a gap equation. In turn this gap equation derives from the restriction of the path integral to Ω by the no pole condition, [1]. In other words the average of $1/\mathcal{M}(A)$ over Ω is finite. This construction radically alters the infrared properties of the theory. For instance, it leads to a gluon propagator which is not fundamental in the sense that it has no (real) pole, [1]. Moreover, it is suppressed in the infrared since it vanishes in the infrared limit. Further, the gap equation implies that the propagator of the Faddeev-Popov ghost is not fundamental but has a dipole behaviour at low momenta which is referred to as ghost enhancement. These infrared properties of the constituent fields are believed to be related to confinement, [1], and over the years has led to intense interest in studying gluon and ghost 2-point functions on the lattice and with Dyson Schwinger equation (DSE) methods.

Another approach was also developed, however, in a series of articles by Zwanziger and collaborators, [2, 3, 4, 5, 6, 7, 8, 9, 10], with other relevant contributions in, for instance, [11, 12]. In essence the semi-classical approach of Gribov for Landau gauge Yang-Mills was put on a firmer footing with the construction of a localized renormalizable Lagrangian, [3, 4, 7, 8]. The renormalizability being established by various authors, [8, 13, 14]. The implementation of the horizon condition defining Ω in the original approach led to a non-local operator in the action which clearly inhibits direct calculations. In [3, 4, 7, 8] Zwanziger localized the non-locality with a (finite) set of extra fields which defined the horizon condition in an equivalent fashion. The beauty of the renormalizability, [8, 13, 14], aside from allowing for calculations was

to demonstrate that *none* of the known and accepted properties of QCD at high energy were changed or upset. For instance, asymptotic freedom remains with the *same* β-function. However, the advantage of the new formulation was to allow for loop calculations and the extension to the next level of computation of the gap equation, gluon suppression and ghost enhancement. This was achieved in [15] and [16]. In the former the two loop $\overline{\mbox{MS}}$ gap equation for γ was established when massless quarks are present. This was checked in a non-trivial way by verifying that ghost enhancement was satisfied at two loops precisely when γ obeyed the gap equation. Indeed the theory has no meaning as a gauge theory unless γ does this and hence is not an independent parameter of the theory, [1]. In the latter article, [16], the one loop gluon suppression was verified as well as the *exact* evaluation of all the one loop 2-point functions of the fields of the Gribov-Zwanziger Lagrangian.

Given this background we come to the main purpose of this article. Clearly in the real world quarks are not massless but massive. Therefore, to have a more realistic understanding of the Gribov situation it seems appropriate to include massive quarks. As will be evident from what is recorded here this is far from a trivial task. First, quarks only appear diagrammatically in the gap equation first at two loops. Moreover, this results in Feynman integrals involving three scales. Aside from the quark mass itself, the gluon propagator actually has two mass scales in the sense of a conventional fundamental propagator. These are $\pm i\sqrt{C_A}\gamma^2$ where the mass is actually imaginary. (The presence of $\sqrt{C_A}$ stems from our conventions which follow those derived in [15, 16].) The presence of the imaginary mass further complicates Feynman integral evaluation since some of the fundamental functions of one and two loop integrals, such as dilogarithms, need to be considered for complex arguments. Therefore, it is the main purpose of this article to extend the massless quark two loop $\overline{\mbox{MS}}$ gap equation of [15] to the massive quark case. Moreover, we will discuss the effect it has on the enhancement of the Faddeev-Popov ghost. Finally, we note that given recent developments concerning the scaling versus decoupling solutions, [17, 18, 19, 20, 21, 22], for which there has yet to be a definitive resolution, we note that our computations will be the foundation for extensions to the decoupling gap equation. This will be required if that solution is eventually established as the correct picture. Moreover, this is possible in our approach because the decoupling solution can be accommodated in the Gribov-Zwanziger formulation, [23, 24]. Though it will in fact be a more difficult task than the current work due to the generation of mass for the localizing Zwanziger ghost fields.

The paper is organised as follows. Section two is devoted to reviewing the relevant aspects of the Gribov-Zwanziger formalism for the massive quark two loop gap equation. The construction of the two loop scalar master integrals to the finite part is presented in section three where we discuss at length their expression in terms of functions of real variables. This is necessary in order to produce a real gap equation rather than a form which has functions of complex variables due to the gluon widths. Our main result is provided in section four whilst we draw our conclusions in section five.

2 Formalism.

In this section we recall the relevant aspects of the basic Gribov-Zwanziger Lagrangian we will use to extend the results of [15]. From [3, 4, 7, 8] the (bare) Lagrangian is

$$\begin{aligned} L^{\mathrm{GZ}} &= L^{\mathrm{QCD}} + \bar{\phi}^{ab\,\mu}\partial^\nu \left(D_\nu \phi_\mu\right)^{ab} - \bar{\omega}^{ab\,\mu}\partial^\nu \left(D_\nu \omega_\mu\right)^{ab} \\ &\quad - g f^{abc}\partial^\nu \bar{\omega}^{ae}_\mu \left(D_\nu c\right)^b \phi^{ec\,\mu} + \frac{\gamma^2}{\sqrt{2}}\left(f^{abc}A^{a\,\mu}\phi^{bc}_\mu - f^{abc}A^{a\,\mu}\bar{\phi}^{bc}_\mu\right) - \frac{dN_A\gamma^4}{2g^2} \end{aligned} \qquad (2.1)$$

where we use the usual linear covariant gauge fixing prescription

$$L^{\mathrm{QCD}} = - \frac{1}{4} G^a_{\mu\nu} G^{a\,\mu\nu} - \frac{1}{2\alpha} (\partial^\mu A^a_\mu)^2 - \bar{c}^a \partial^\mu D_\mu c^a + i \bar{\psi}^{iI} \not{D} \psi^{iI} - m_q \bar{\psi}^{iI} \psi^{iI} . \quad (2.2)$$

Although we will work strictly in the Landau gauge we have included the usual gauge fixing parameter α since it is required to derive the gluon propagator. Aside from this it should be understood that α is set to zero throughout. Briefly our conventions in (2.1) and (2.2) are that A^a_μ is the gluon, c^a is the Faddeev-Popov ghost, ψ^{iI} is the quark with mass m_q and ϕ^{ab}_μ, $\bar{\phi}^{ab}_\mu$, ω^{ab}_μ and $\bar{\omega}^{ab}_\mu$ are the Zwanziger localizing ghosts. The latter pair are anti-commuting like the Faddeev-Popov ghosts whereas ϕ^{ab}_μ and $\bar{\phi}^{ab}_\mu$ are commuting. The Lagrangian is expressed in d-dimensional spacetime since we will use dimensional regularization throughout to isolate the divergence structure of the Feynman graphs where $d = 4 - 2\epsilon$ and ϵ is the regularizing parameter. The various indices have the ranges $1 \leq I \leq N_f$, $1 \leq a \leq N_A$ and $1 \leq i \leq N_F$ where N_f is the number of quark flavours and N_F and N_A are the respective dimensions of the fundamental and adjoint representations. The various covariant derivatives are

$$\begin{aligned} D_\mu c^a &= \partial_\mu c^a - g f^{abc} A^b_\mu c^c \\ D_\mu \psi^{iI} &= \partial_\mu \psi^{iI} + i g T^a_{IJ} A^a_\mu \psi^{iJ} \\ (D_\mu \phi_\nu)^{ab} &= \partial_\mu \phi^{ab}_\nu - g f^{acd} A^c_\mu \phi^{db}_\nu \end{aligned} \quad (2.3)$$

where g is the coupling constant, $G^a_{\mu\nu}$ is the usual gluon field strength and T^a are the generators of the colour group which has structure functions f^{abc}. We note that we have reverted to the conventions of the original form of the Lagrangian, [4, 8], in the mixed 2-point sector*. With this formulation of the Gribov-Zwanziger Lagrangian we have checked that the results of the massless quark gap equation at two loops and Faddeev-Popov ghost enhancement correctly emerge. The fields ϕ^{ab}_μ and $\bar{\phi}^{ab}_\mu$ correspond to the localization of the Gribov horizon condition which originally was

$$\left\langle A^a_\mu(x) \frac{1}{\partial^\nu D_\nu} A^{a\,\mu}(x) \right\rangle = \frac{d N_A}{C_A g^2} \quad (2.4)$$

and now equates to

$$\begin{aligned} f^{abc} \langle A^{a\,\mu}(x) \phi^{bc}_\mu(x) \rangle &= \frac{d N_A \gamma^2}{\sqrt{2} g^2} \\ f^{abc} \langle A^{a\,\mu}(x) \bar{\phi}^{bc}_\mu(x) \rangle &= - \frac{d N_A \gamma^2}{\sqrt{2} g^2} . \end{aligned} \quad (2.5)$$

Our conventions are actually crucial to reproducing the correct form of the gluon propagator of the original Gribov article, [1]. Using other conventions could lead to, for example, a gluon propagator which has a normal mass as well as a tachyonic mass. From (2.1) and (2.2) we have checked that the propagators of the fields, with momentum p, are

$$\begin{aligned} \langle A^a_\mu(p) A^b_\nu(-p) \rangle &= - \frac{\delta^{ab} p^2}{[(p^2)^2 + C_A \gamma^4]} P_{\mu\nu}(p) \\ \langle A^a_\mu(p) \bar{\phi}^{bc}_\nu(-p) \rangle &= - \frac{f^{abc} \gamma^2}{\sqrt{2}[(p^2)^2 + C_A \gamma^4]} P_{\mu\nu}(p) \\ \langle \phi^{ab}_\mu(p) \bar{\phi}^{cd}_\nu(-p) \rangle &= - \frac{\delta^{ac} \delta^{bd}}{p^2} \eta_{\mu\nu} + \frac{f^{abe} f^{cde} \gamma^4}{p^2 [(p^2)^2 + C_A \gamma^4]} P_{\mu\nu}(p) \end{aligned}$$

*In [15, 16] the Feynman rules of this Lagrangian were used within the computer algebra computations though the actual Lagrangian recorded in the articles followed the conventions of [14].

$$\begin{aligned}
\langle \omega^{ab}_\mu(p) \bar{\omega}^{cd}_\nu(-p) \rangle &= - \frac{\delta^{ac}\delta^{bd}}{p^2} \eta_{\mu\nu} \\
\langle c^a(p) \bar{c}^b(-p) \rangle &= \frac{\delta^{ab}}{p^2} \\
\langle \psi^{iI}(p) \bar{\psi}^{jJ}(-p) \rangle &= \delta^{ij} \delta^{IJ} \frac{(\not{p} + m_q)}{[p^2 + m_q^2]}
\end{aligned} \tag{2.6}$$

in the Landau gauge where

$$P_{\mu\nu}(p) ~=~ \eta_{\mu\nu} ~-~ \frac{p_\mu p_\nu}{p^2} \tag{2.7}$$

is the usual projector. We have retained a non-zero α in inverting the matrix of 2-point functions in the quadratic part of the Lagrangian before setting $\alpha = 0$ to recover the Landau gauge. The Feynman rules for the vertices have no convention complications and are straightforward to derive from (2.1) and (2.2). Though we note that the explicit cubic interaction of (2.1) is completely passive since it is never present within the Feynman diagrams contributing to any Green's function of interest at the two loop level of this article.

As (2.1) is renormalizable and incorporates the Gribov properties we now discuss the set-up for our computation. The gap equation satisfied by γ is defined by the no-pole condition determining the boundary of Ω, [1]. In the original approach of [1] this equated to evaluating the vacuum expectation value of $f^{abc} A^{a\,\mu} \phi^{ab}_\mu$ and ensuring it satisfied (2.5) where the right side is a finite object, [3, 4, 7, 8]. However, at this point we note that in all the vacuum expectation values one has to take into account the renormalization of the fields and parameters. In this respect we note that the anomalous dimensions of all the quantities we require are available at three loops in the $\overline{\mathrm{MS}}$ scheme for an arbitrary colour group, [13, 14, 15, 16]. Further, at four loops the renormalization of γ is known for the $SU(N_c)$ Lie colour groups, [25]. These follow partly through the renormalizability of (2.1), [8, 13, 14], but also because the localizing fields and γ do not undergo independent renormalization in the Landau gauge. Instead all the renormalization constants are determined by Slavnov-Taylor identities, [8, 13, 14]. Denoting the associated anomalous dimensions of a field or parameter Γ by $\gamma_\Gamma(a)$ where $a = g^2/(16\pi^2)$ then we record the renormalization constants we require as being encoded in the anomalous dimensions

$$\begin{aligned}
\gamma_A(a) &= [8T_F N_f - 13C_A] \frac{a}{6} ~+~ \left[40C_A T_F N_f + 32C_F T_F N_f - 59C_A^2 \right] \frac{a^2}{8} ~+~ O(a^3) \\
\gamma_\phi(a) &= \gamma_\omega(a) ~=~ -~ \frac{3}{4} C_A a ~+~ \left[40C_A T_F N_f - 95C_A^2 \right] \frac{a^2}{48} ~+~ O(a^3) \\
\gamma_\gamma(a) &= [16T_F N_f - 35C_A] \frac{a}{48} ~+~ \left[280C_A T_F N_f - 449C_A^2 + 192C_F T_F N_f \right] \frac{a^2}{192} ~+~ O(a^3)
\end{aligned} \tag{2.8}$$

with the β-function

$$\beta(a) ~=~ -~ \left[\frac{11}{3} C_A - \frac{4}{3} T_F N_f \right] a^2 ~-~ \left[\frac{34}{3} C_A^2 - 4C_F T_F N_f - \frac{20}{3} C_A T_F N_f \right] a^3 ~+~ O(a^4) ~. \tag{2.9}$$

The elementary group Casimirs are defined by

$$\mathrm{Tr} \left(T^a T^b \right) ~=~ T_F \delta^{ab} ~~,~~ T^a T^a ~=~ C_F I ~~,~~ f^{acd} f^{bcd} ~=~ C_A \delta^{ab} ~. \tag{2.10}$$

Whilst the higher order expressions are available we only provide them at two loops as that is the order we compute to here. In (2.5) we note that the renormalization of all fields and parameters present is therefore already fixed and hence after all contributing Feynman diagrams have been

computed and assembled the resulting vacuum expectation value is finite. With a massive quark present, its mass will be renormalized in principle too. However, as it first appears at two loops, scaling it from a bare to a renormalized parameter will not affect the two loop gap calculation as the counterterms from the quark mass renormalization constant will only arise first at three loops.

As (2.5) is the vacuum expectation value of two fields it is easy to determine since essentially it is the closure of the legs on the mixed propagator of (2.6) and integrated over the momentum p. Thus for higher loop calculations one simply evaluates the relevant Feynman diagrams which are merely vacuum bubbles with various configurations of masses. We devolve to a later section the more detailed structure of such two loop massive vacuum bubbles and concentrate in the remainder of this section on more general aspects of the two loop gap equation calculation. The main ingredients are the generation of the Feynman graphs via the QGRAF package, [26], and its conversion into the symbolic manipulation language FORM, [27]. We use FORM as it is ideal for handling the underlying algebra in an efficient manner. For the gap equation, due to the mixed propagators, there are 1 one loop and 17 two loop Feynman diagrams to be determined exactly as a function of γ and m_q. As they resolve into the basic structure of two loop vacuum bubbles, we note that to make contact with known results we apply elementary partial fractions to the common factor in the propagators of the Gribov related fields, such as

$$\frac{p^2}{[(p^2)^2 + C_A\gamma^4]} = \frac{1}{2}\left(\frac{1}{[p^2 + i\sqrt{C_A}\gamma^2]} + \frac{1}{[p^2 - i\sqrt{C_A}\gamma^2]} \right) . \tag{2.11}$$

Moreover, within our FORM routines the Feynman rules are substituted automatically and the elementary group theory is evaluated making extensive use of the Jacobi identity for the structure functions, partly due to the form of the pure ϕ^{ab}_μ propagator. Therefore, all that remains in determining the gap equation for massive quarks is the substitution of the explicit forms for the master Feynman integrals which the FORM routines produce. The next section is devoted to this where we concentrate on the intricacies of dealing with vaccum integrals with massive quarks and complex gluon masses.

3 Master integrals.

There are two main scalar master integrals which arise in the computation. The first is the massive one loop vacuum bubble which is virtually trivial in comparison with that we have to consider at two loops. Though it does arise in the two loop computation when a line in the basic form of a two loop vacuum bubble graph is omitted. Therefore, defining

$$I_1(m^2) = \int_k \frac{1}{[k^2 + m^2]} \tag{3.1}$$

where

$$\int_k \equiv \int \frac{d^dk}{(2\pi)^d} \tag{3.2}$$

includes the d-dimensional momentum space measure we have exactly

$$I_1(m^2) = \frac{\Gamma(1 - \frac{1}{2}d)}{(4\pi)^{d/2}} (m^2)^{\frac{1}{2}d - 1} \tag{3.3}$$

which is trivial to expand in powers of ϵ. Therefore, we now concentrate on the basic massive scalar two loop vacuum bubble which we define as

$$I_2(m_x^2, m_y^2, m_z^2) = \int_k \int_l \frac{1}{[k^2 + m_x^2]\left[(k - l)^2 + m_y^2\right][l^2 + m_z^2]} . \tag{3.4}$$

which is completely symmetric in its arguments and has been studied extensively over the years. See, for example, [28, 29, 30]. Its expansion in powers of ϵ, where $d = 4 - 2\epsilon$, is known to several orders but for our purposes it suffices to record it to the finite part. For this we follow the notation and conventions of [29]. Then we have

$$\begin{aligned}(4\pi)^4 I_2(x,y,z) &= -\frac{c}{2\epsilon^2} - \frac{1}{\epsilon}\left[\frac{3c}{2} - \mathrm{L}_1\right] \\ &\quad - \frac{1}{2}\Big[L_2 - 6L_1 + \xi(x,y,z) + c\,(7+\zeta(2)) \\ &\qquad + (y+z-x)\,\overline{\ln}(y)\overline{\ln}(z) + (z+x-y)\,\overline{\ln}(z)\overline{\ln}(y) \\ &\qquad + (y+x-z)\,\overline{\ln}(y)\overline{\ln}(x)\Big] + O(\epsilon) \end{aligned} \tag{3.5}$$

where we define

$$\begin{aligned} L_i &= x\overline{\ln}^i(x) + y\overline{\ln}^i(y) + z\overline{\ln}^i(z) \\ c &= x + y + z \\ a &= \frac{1}{2}\left[x^2+y^2+z^2-2xy-2xz-2yz\right]^{1/2} . \end{aligned} \tag{3.6}$$

We also use the same notation as [29] in defining

$$\overline{\ln}(m^2) = \ln\left(\frac{m^2}{\mu^2}\right) \tag{3.7}$$

where μ is the mass scale which enters when using dimensional regularization to ensure the coupling constant remains dimensionless in d-dimensions. The key part of this ϵ expansion is the function $\xi(x,y,z)$ whose explicit form depends on the sign of the combination of masses denoted by a^2. For our purposes we note that for $a^2 > 0$ then, [29],

$$\xi(x,y,z) = 8a\,[M(\phi_z) + M(\phi_y) - M(-\phi_x)] \tag{3.8}$$

where

$$M(\phi) = -\int_0^\phi d\theta \ln(\sinh(\theta)) \tag{3.9}$$

and

$$\phi_x = \coth^{-1}\left[\frac{c-2x}{2a}\right] . \tag{3.10}$$

Moreover, we note that

$$\coth^{-1}(z) = \frac{1}{2}\ln\left[\frac{z+1}{z-1}\right] \tag{3.11}$$

and the integral defined by the intermediate function $M(\phi)$ can be written in terms of known functions

$$M(\phi) = \phi\ln(2) - \tfrac{1}{2}\phi^2 + \tfrac{1}{2}\zeta(2) - \mathrm{Li}_2(e^{-\phi}) - \mathrm{Li}_2(-e^{-\phi}) \tag{3.12}$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function, [31],

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-x)}{x}dx \tag{3.13}$$

and $\zeta(z)$ is the Riemann zeta function. As an exercise to aid the interested reader it is instructive to consider the elementary case $I_2(0,0,m^2)$ which is explicitly

$$I_2(0,0,m^2) = \int_k\int_l \frac{1}{k^2(k-l)^2\,[l^2+m^2]} . \tag{3.14}$$

It can be evaluated directly and then compared with (3.5) to give

$$I_2(0,0,m^2) \;=\; -\;\frac{m^2}{2\epsilon^2} \;-\; \frac{m^2}{2\epsilon}\left[3-2\overline{\ln}(m^2)\right] \;-\; \frac{m^2}{2}\left[7+3\zeta(2)+2\overline{\ln}^2(m^2)-6\overline{\ln}(m^2)\right] \;+\; O(\epsilon) \tag{3.15}$$

which will be required for checking our expressions in the massless quark limit.

However, as we are ultimately interested in the massive quark case, we have to consider several master integrals. These are

$$\begin{aligned} I_2(m_q^2,m_q^2,i\sqrt{C_A}\gamma^2) &= \int_k\int_l \frac{1}{\left[k^2+m_q^2\right]\left[(k-l)^2+m_q^2\right][l^2+i\sqrt{C_A}\gamma^2]} \\ I_2(m_q^2,m_q^2,-i\sqrt{C_A}\gamma^2) &= \int_k\int_l \frac{1}{\left[k^2+m_q^2\right]\left[(k-l)^2+m_q^2\right][l^2-i\sqrt{C_A}\gamma^2]} \end{aligned} \tag{3.16}$$

and the related integrals

$$\begin{aligned} \bar{I}_2(m_q^2,m_q^2,i\sqrt{C_A}\gamma^2) &= \int_k\int_l \frac{1}{\left[k^2+m_q^2\right]\left[(k-l)^2+m_q^2\right][l^2+i\sqrt{C_A}\gamma^2]^2} \\ \bar{I}_2(m_q^2,m_q^2,-i\sqrt{C_A}\gamma^2) &= \int_k\int_l \frac{1}{\left[k^2+m_q^2\right]\left[(k-l)^2+m_q^2\right][l^2-i\sqrt{C_A}\gamma^2]^2} \;. \end{aligned} \tag{3.17}$$

We concentrate on the former two as the definition of the latter follows from using elementary calculus. For the first we will focus on

$$\xi(i\sqrt{C_A}\gamma^2,m_q^2,m_q^2) \;=\; 8a\left[2M(\phi_{m_q^2})-M(-\phi_{i\sqrt{C_A}\gamma^2})\right] \tag{3.18}$$

where now

$$a \;=\; \frac{i}{2}\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2m_q^2} \quad , \quad c \;=\; i\sqrt{C_A}\gamma^2 \;+\; 2m_q^2 \;. \tag{3.19}$$

leading to the intermediate variables

$$\begin{aligned} \phi_{i\sqrt{C_A}\gamma^2} &= \coth^{-1}\left[\frac{-\sqrt{C_A}\gamma^2-2im_q^2}{\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2m_q^2}}\right] \\ \phi_{m_q^2} &= \coth^{-1}\left[\frac{\sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2m_q^2}}\right] \;. \end{aligned} \tag{3.20}$$

Though for practical purposes it is more appropriate to re-express these by applying the logarithm definition

$$\begin{aligned} e^{-\phi_{i\sqrt{C_A}\gamma^2}} &= \frac{\sqrt{C_A}\gamma^2+\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2m_q^2}}{\sqrt{C_A}\gamma^2-\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2m_q^2}} \\ e^{-\phi_{m_q^2}} &= \sqrt{\frac{\sqrt{C_A}\gamma^2-\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2m_q^2}}{\sqrt{C_A}\gamma^2+\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2m_q^2}}} \;. \end{aligned} \tag{3.21}$$

These naturally lead to the two functions

$$M(t_{m_q^2}) \;=\; \frac{\zeta(2)}{2} \;+\; \frac{1}{2}\ln\left[\frac{\sqrt{C_A}\gamma^2+\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2m_q^2}}{\sqrt{C_A}\gamma^2-\sqrt{C_A\gamma^4+4i\sqrt{C_A}\gamma^2m_q^2}}\right]\ln(2)$$

$$
\begin{aligned}
& - \frac{1}{8} \ln^2 \left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \\
& - \frac{1}{2} \mathrm{Li}_2 \left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right]
\end{aligned} \tag{3.22}
$$

and

$$
\begin{aligned}
M(-\phi_{i\sqrt{C_A}\gamma^2}) \;=\; & \frac{\zeta(2)}{2} \;+\; \ln \left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \ln(2) \\
& - \frac{1}{2} \ln^2 \left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \\
& - \mathrm{Li}_2 \left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \\
& - \mathrm{Li}_2 \left[\frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right]
\end{aligned} \tag{3.23}
$$

where we have used the relationship, [31],

$$
\mathrm{Li}_2(x) \;+\; \mathrm{Li}_2(-x) \;=\; \frac{1}{2}\mathrm{Li}_2(x^2) \; . \tag{3.24}
$$

Remarkably, this leads to the compact expression

$$
\begin{aligned}
\xi(i\sqrt{C_A}\gamma^2, m_q^2, m_q^2) \;=\; & 4i\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} \\
& \times \left[\frac{\zeta(2)}{2} \;+\; \frac{1}{4} \ln^2 \left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \right. \\
& \left. \;+\; \mathrm{Li}_2 \left[\frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \right]
\end{aligned} \tag{3.25}
$$

giving the integral to the finite part

$$
\begin{aligned}
I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) \;=\; & - \frac{1}{2\epsilon^2} \left(i\sqrt{C_A}\gamma^2 \;+\; 2m_q^2 \right) \\
& - \frac{1}{\epsilon} \left(\frac{1}{2}(3i\sqrt{C_A}\gamma^2 + 6m_q^2) \;-\; 2m_q^2 \overline{\ln}(m_q^2) \;-\; i\sqrt{C_A}\gamma^2 \overline{\ln}(i\sqrt{C_A}\gamma^2) \right) \\
& - m_q^2 \overline{\ln}^2(m_q^2) \;-\; \frac{1}{2} i\sqrt{C_A}\gamma^2 \overline{\ln}^2(i\sqrt{C_A}\gamma^2) \\
& + 6m_q^2 \overline{\ln}(m_q^2) \;+\; 3i\sqrt{C_A}\gamma^2 \overline{\ln}(i\sqrt{C_A}\gamma^2) \\
& - 2i\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} \\
& \times \left[\frac{\zeta(2)}{2} \;+\; \frac{1}{4} \ln^2 \left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \right.
\end{aligned}
$$

$$
\begin{aligned}
& + \ \mathrm{Li}_2 \left[\frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2} \right] \Bigg] \\
& - \frac{1}{2} \left(i\sqrt{C_A}\gamma^2 \ + \ 2m_q^2 \right) (7 + \zeta(2)) \\
& - \frac{1}{2} \left(2m_q^2 - i\sqrt{C_A}\gamma^2 \right) \overline{\ln}(m_q^2) \\
& - \ i\sqrt{C_A}\gamma^2 \overline{\ln}(m_q^2) \overline{\ln}(i\sqrt{C_A}\gamma^2) \ + \ O(\epsilon) \ .
\end{aligned} \tag{3.26}
$$

We have checked that this expression correctly reduces to that for $I_2(0,0,i\sqrt{C_A}\gamma^2)$ in the limit $m_q^2 \to 0$. This is not as straightforward as it seems due to the presence of the dilogarithm function and terms involving $\overline{\ln}(m_q^2)$. Disregarding all terms proportional to m_q^2 and expanding $\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}$ in powers of m_q^2, then $I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2)$ reduces to

$$
\begin{aligned}
I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2) \ = \ & - \ \frac{i\sqrt{C_A}\gamma^2}{2\epsilon^2} \ - \ \frac{i\sqrt{C_A}\gamma^2}{\epsilon} \left(\frac{3}{2} - \overline{\ln}(i\sqrt{C_A}\gamma^2) \right) \\
& - \frac{1}{2} i\sqrt{C_A}\gamma^2 \overline{\ln}^2(i\sqrt{C_A}\gamma^2) \ + \ 3i\sqrt{C_A}\gamma^2 \overline{\ln}(i\sqrt{C_A}\gamma^2) \\
& - i\sqrt{C_A}\gamma^2 \Bigg[\zeta(2) \ + \ 2\mathrm{Li}_2 \left[\frac{i m_q^2}{\sqrt{C_A}\gamma^2} \right] \\
& \qquad + \ \frac{1}{2} \left[\overline{\ln}^2(i\sqrt{C_A}\gamma^2) - 2\overline{\ln}(i\sqrt{C_A}\gamma^2)\overline{\ln}(m_q^2) + \overline{\ln}^2(m_q^2) \right] \Bigg] \\
& - \ \frac{i\sqrt{C_A}\gamma^2}{2} (7 + \zeta(2)) \ + \ \frac{i\sqrt{C_A}\gamma^2}{2} \overline{\ln}^2(m_q^2) \\
& - i\sqrt{C_A}\gamma^2 \overline{\ln}(m_q^2) \overline{\ln}(i\sqrt{C_A}\gamma^2) \ + \ O(m_q^2; \epsilon)
\end{aligned} \tag{3.27}
$$

where we note that the imaginary dilogarithm vanishes as $m_q^2 \to 0$ and the remaining logarithmic terms in m_q^2 cancel. By making the analytic continuation $m^2 \to i\sqrt{C_A}\gamma^2$ in (3.15), we see that our integral $I_2(m_q^2, m_q^2, i\sqrt{C_A}\gamma^2)$ is entirely consistent with $I_2(0,0,i\sqrt{C_A}\gamma^2)$ in the limit of zero quark mass.

Next we turn to the complex conjugate integral and focus on

$$
\xi(-i\sqrt{C_A}\gamma^2, m_q^2, m_q^2) \ = \ 8a \left[2M(\phi_{m_q^2}) - M(-\phi_{-i\sqrt{C_A}\gamma^2}) \right] \tag{3.28}
$$

where now the variables are

$$
a \ = \ \frac{i}{2} \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} \quad , \quad c \ = \ 2m_q^2 - i\sqrt{C_A}\gamma^2 \tag{3.29}
$$

leading to

$$
\begin{aligned}
e^{-\phi_{i\sqrt{C_A}\gamma^2}} \ &= \ \frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \\
e^{-\phi_{m_q^2}} \ &= \ \sqrt{\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}} \ .
\end{aligned} \tag{3.30}
$$

Without reproducing analogous manipulations, we find

$$
\xi(-i\sqrt{C_A}\gamma^2, m_q^2, m_q^2) \ = \ 4i\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}
$$

$$
\times \left[\frac{\zeta(2)}{2} ~+~ \frac{1}{4} \ln^2 \left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \right.
$$

$$
\left. +~ \mathrm{Li}_2 \left[\frac{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2} \right] \right] ~. \tag{3.31}
$$

Whilst this is similar to $\xi(i\sqrt{C_A}\gamma^2, m_q^2, m_q^2)$ there is a potential singularity in the massless quark limit arising from the dilogarithm term. To circumvent this and to have a final expression for the integral $I_2(-i\sqrt{C_A}\gamma^2, m_q^2, m_q^2)$ which is clearly the complex conjugate of $I_2(i\sqrt{C_A}\gamma^2, m_q^2, m_q^2)$ we use the dilogarithm identity, [31],

$$
\mathrm{Li}_2(-1/z) ~+~ \mathrm{Li}_2(-z) ~~=~~ -\,\zeta(2) ~-~ \frac{1}{2}\ln^2(z) \tag{3.32}
$$

with

$$
z ~=~ -~ \left[\frac{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2} \right] ~. \tag{3.33}
$$

Given this we end up with the final expression

$$
\begin{aligned}
I_2(-i\sqrt{C_A}\gamma^2, m_q^2, m_q^2) ~~=~~ & -~ \frac{1}{2\epsilon^2} \left(2m_q^2 ~-~ i\sqrt{C_A}\gamma^2 \right) \\
& -~ \frac{1}{\epsilon} \left[\frac{1}{2}(6m_q^2 - 3i\sqrt{C_A}\gamma^2) - 2m_q^2 \overline{\ln}(m_q^2) + i\sqrt{C_A}\gamma^2 \overline{\ln}(-i\sqrt{C_A}\gamma^2) \right] \\
& -~ m_q^2 (\overline{\ln}(m_q^2))^2 ~+~ \frac{1}{2} i\sqrt{C_A}\gamma^2 (\overline{\ln}(-i\sqrt{C_A}\gamma^2))^2 \\
& +~ 6m_q^2 \overline{\ln}(m_q^2) ~-~ 3i\sqrt{C_A}\gamma^2 \overline{\ln}(-i\sqrt{C_A}\gamma^2) \\
& +~ 2i\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} \\
& \qquad \times \left[\frac{\zeta(2)}{2} ~+~ \frac{1}{4}\ln^2 \left[\frac{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \right. \\
& \qquad\quad \left. +~ \mathrm{Li}_2 \left[\frac{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2} \right] \right] \\
& -~ \frac{1}{2} \left(2m_q^2 ~-~ i\sqrt{C_A}\gamma^2 \right) (7 + \zeta(2)) \\
& -~ \frac{1}{2} \left(2m_q^2 + i\sqrt{C_A}\gamma^2 \right) \overline{\ln}^2(m_q^2) \\
& +~ i\sqrt{C_A}\gamma^2 \overline{\ln}(m_q^2) \overline{\ln}(-i\sqrt{C_A}\gamma^2) ~+~ O(\epsilon) ~.
\end{aligned} \tag{3.34}
$$

Comparing this with our expression, (3.26), we see that the explicit forms of $I_2(i\sqrt{C_A}\gamma^2, m_q^2, m_q^2)$ and $I_2(-i\sqrt{C_A}\gamma^2, m_q^2, m_q^2)$ are indeed complex conjugates as expected from their original definitions. This is an important check on our manipulations and use of dilogarithm identities and ensure that the correct massless quark limits will emerge which is important for checking our eventual gap equation.

The remaining two master integrals, (3.17), can be simply deduced from the above expressions by differentiating with respect to γ^2. As this is elementary we merely note the explicit expression for the first of (3.17) is

$$
\bar{I}_2(i\sqrt{C_A}\gamma^2, m_q^2, m_q^2) ~~=~~ \frac{1}{2\epsilon^2} ~-~ \frac{1}{\epsilon} \left(\overline{\ln}(i\sqrt{C_A}\gamma^2) - \frac{1}{2} \right)
$$

$$
\begin{aligned}
&+ \frac{1}{2}\overline{\ln}^2(i\sqrt{C_A}\gamma^2) \; - \; 5\overline{\ln}(i\sqrt{C_A}\gamma^2) + \frac{1}{2} + \frac{\zeta(2)}{2} \; + \; 2i\pi \\
&- \frac{1}{2}\overline{\ln}^2(m_q^2) \; + \; \overline{\ln}(m_q^2)\overline{\ln}(i\sqrt{C_A}\gamma^2) \; - \; 4\ln(2) \\
&+ \left[\frac{\left(2\sqrt{C_A}\gamma^2 + 4im_q^2\right)\sqrt{C_A\gamma^4 - 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A^2\gamma^8 + 16\sqrt{C_A}^2\gamma^4 m_q^4}} \right] \\
&\quad \times \left[\frac{\zeta(2)}{2} \; + \; \frac{1}{4}\ln^2\left[\frac{\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}}{\sqrt{C_A}\gamma^2 - \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2}} \right] \right. \\
&\qquad \left. + \; \mathrm{Li}_2\left[\frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2} \right] \right] \\
&\qquad + \; 4\overline{\ln}\left[\sqrt{C_A}\gamma^2 + \sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} \right] \; + \; O(\epsilon) \qquad (3.35)
\end{aligned}
$$

where we have used

$$
\frac{d}{dz}\mathrm{Li}_2(z) \;=\; - \; \frac{\ln(1-z)}{z} \; . \qquad (3.36)
$$

Again we have checked that the correct massless quark limit emerges with the direct evaluation of the equivalent integral.

Whilst we have now determined all the master integrals to the finite part in the ϵ expansion, the explicit expressions are not in a fully useful format. Given that the ultimate gap equation is a real function we need to write the expressions as a real and imaginary part. This is not a simple exercise due to the presence of the dilogarithm of a complex argument. However, the theory behind such functions is known, [31], and we summarize what we require for the current calculation. Writing the complex variable z in polar form we have the real and imaginary parts, [31],

$$
\mathrm{Li}_2(re^{i\theta}) \;=\; \mathrm{Li}_2(r,\theta) \;+\; i\left[\omega\ln(r) + \tfrac{1}{2}\mathrm{Cl}_2(2\omega) + \tfrac{1}{2}\mathrm{Cl}_2(2\theta) - \tfrac{1}{2}\mathrm{Cl}_2(2\omega + 2\theta)\right] \qquad (3.37)
$$

where

$$
\mathrm{Li}_2(r,\theta) \;=\; - \; \frac{1}{2}\int_0^r \frac{\ln(1 - 2x\cos\theta + x^2)}{x} dx \qquad (3.38)
$$

and $\mathrm{Cl}_2(\theta)$ is the Clausen function defined by

$$
\mathrm{Cl}_2(\theta) \;=\; - \; \int_0^\theta \ln\left[2\sin\left(\frac{\phi}{2}\right)\right] d\phi \; . \qquad (3.39)
$$

The intermediate angle ω is related to the polar variables r and θ of z by

$$
\omega \;=\; \tan^{-1}\left(\frac{r\sin\theta}{1 - r\cos\theta}\right) \; . \qquad (3.40)
$$

Given these general definitions then to proceed with our simplification to real and imaginary parts, we need to write the arguments of the dilogarithms in polar forms. To assist this we recall the elementary lemma for a complex variable $z = a + ib$, where a and b are real,

$$
\sqrt{a \pm ib} \;=\; \frac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2} + a} \;\pm\; \frac{i}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2} - a} \; . \qquad (3.41)
$$

So, for example,

$$\begin{aligned}\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} &= \frac{1}{\sqrt{2}}\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} + C_A\gamma^4} \\ &\quad + \frac{i}{\sqrt{2}}\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} - C_A\gamma^4} \, . \end{aligned} \tag{3.42}$$

For the dilogarithms if we set

$$re^{i\theta} \equiv \frac{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} - \sqrt{C_A}\gamma^2}{\sqrt{C_A\gamma^4 + 4i\sqrt{C_A}\gamma^2 m_q^2} + \sqrt{C_A}\gamma^2} \tag{3.43}$$

then

$$re^{i\theta} = \frac{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} - C_A\gamma^4 + i\sqrt{2}\sqrt{C_A}\gamma^2\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} - C_A\gamma^4}}{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} + C_A\gamma^4 + \sqrt{2}\sqrt{C_A}\gamma^2\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} + C_A\gamma^4}} \, . \tag{3.44}$$

giving

$$\begin{aligned} r &= \frac{4\sqrt{C_A}\gamma^2 m_q^2}{\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} + C_A\gamma^4}\left(\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} + C_A\gamma^4}\right)} \\ \tan\theta &= \frac{\sqrt{2}\sqrt{C_A}\gamma^2}{\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} - C_A\gamma^4}} \, . \end{aligned} \tag{3.45}$$

In what follows we will always regard r and θ as taking these values with the associated corresponding value of ω. Although the dilogarithm is the most involved of the terms which appear in the finite parts, similar manipulation is required for several of the logarithm terms. Collecting all the pieces together we find the following expression written as real and imaginary parts,

$$\begin{aligned} I_2(i\sqrt{C_A}\gamma^2, m_q^2, m_q^2) &= -\frac{1}{2\epsilon^2}\left(i\sqrt{C_A}\gamma^2 \;+\; 2m_q^2\right) \\ &\quad - \frac{1}{\epsilon}\left(\frac{1}{2}(3i\sqrt{C_A}\gamma^2 + 6m_q^2) - 2m_q^2\overline{\ln}(m_q^2) - i\sqrt{C_A}\gamma^2\overline{\ln}(i\sqrt{C_A}\gamma^2)\right) \\ &\quad - m_q^2\overline{\ln}^2(m_q^2) \;-\; \frac{1}{2}i\sqrt{C_A}\gamma^2\overline{\ln}^2(i\sqrt{C_A}\gamma^2) \\ &\quad + 6m_q^2\overline{\ln}(m_q^2) \;+\; 3i\sqrt{C_A}\gamma^2\overline{\ln}(i\sqrt{C_A}\gamma^2) \\ &\quad - \sqrt{2}i\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} + C_A\gamma^4} \\ &\qquad \times \left[\frac{1}{4}\left[\frac{1}{2}\overline{\ln}\left(\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4\right)\right.\right. \\ &\qquad\qquad \times \overline{\ln}\left(\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right) \\ &\qquad\qquad + \; 2i\tan^{-1}\left[\frac{\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} - C_A\gamma^4}}{\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}}\right] \\ &\qquad\qquad \left.\left. - \; 2\ln(2) + \frac{1}{2}i\pi - \overline{\ln}(\sqrt{C_A}\gamma^2) - \overline{\ln}(m_q^2) + \frac{\zeta(2)}{2}\right]^2 \right. \end{aligned}$$

$$
\begin{aligned}
& + \ \mathrm{Li}_2(r,\theta) \\
& + \ i\omega \left[2\ln(2) + \frac{1}{2}\overline{\ln}(C_A\gamma^4) + \overline{\ln}(m_q^2) \right. \\
& \qquad - \ \frac{1}{2}\overline{\ln}\left(\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4\right) \\
& \qquad \left. - \ \overline{\ln}\left(\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right)\right] \\
& + \ \frac{i}{2}\mathrm{Cl}_2(2\omega) + \frac{i}{2}\mathrm{Cl}_2(2\theta) - \frac{i}{2}\mathrm{Cl}_2(2\omega + 2\theta)\Big] \\
+ \ & \sqrt{2}\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} - C_A\gamma^4} \\
\times & \left[\frac{1}{4}\left[\frac{1}{2}\overline{\ln}\left(\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4\right)\right.\right. \\
& \quad \times \overline{\ln}\left(\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right) \\
& \quad + \ 2i\tan^{-1}\left[\frac{\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} - C_A\gamma^4}}{\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}}\right] \\
& \quad \left. - \ 2\ln(2) + \frac{1}{2}i\pi - \overline{\ln}(\sqrt{C_A}\gamma^2) - \overline{\ln}(m_q^2) + \frac{\zeta(2)}{2}\right]^2 \\
& + \ \mathrm{Li}_2(r,\theta) \\
& + \ i\omega \left[2\ln(2) + \frac{1}{2}\overline{\ln}(C_A\gamma^4) + \overline{\ln}(m_q^2) \right. \\
& \qquad - \ \frac{1}{2}\overline{\ln}\left(\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4\right) \\
& \qquad \left. - \ \overline{\ln}\left(\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right)\right] \\
& + \ \frac{i}{2}\mathrm{Cl}_2(2\omega) + \frac{i}{2}\mathrm{Cl}_2(2\theta) - \frac{i}{2}\mathrm{Cl}_2(2\omega + 2\theta)\Big] \\
- \ & \frac{1}{2}\left(i\sqrt{C_A}\gamma^2 \ + \ 2m_q^2\right)(7 + \zeta(2)) \ - \ \frac{1}{2}\left(2m_q^2 - i\sqrt{C_A}\gamma^2\right)\overline{\ln}^2(m_q^2) \\
- \ & i\sqrt{C_A}\gamma^2\overline{\ln}(m_q^2)\overline{\ln}(i\sqrt{C_A}\gamma^2) \ + \ O(\epsilon) \ .
\end{aligned}
\tag{3.46}
$$

Whilst this is not truly of the form $a + ib$ since not all terms have been fully multiplied out and there are logarithms with purely imaginary arguments, we prefer to leave it in this more compact form since, for instance, it is elementary to implement

$$
\overline{\ln}(i\sqrt{C_A}\gamma^2) \ = \ \overline{\ln}(\sqrt{C_A}\gamma^2) \ + \ \frac{i\pi}{2}
\tag{3.47}
$$

within our FORM routines. This also takes care of the other elementary complex algebra automatically.

4 Two loop gap equation.

Equipped with the basic master integrals we are now in a position to assemble the two loop Gribov gap equation in the $\overline{\mathrm{MS}}$ scheme with massive quarks. This requires the evaluation of

the seventeen contributing Feynman diagrams which without the power of FORM would have been virtually impossible. Having already discussed the key aspects of the computation, we ultimately find

$$
\begin{aligned}
1 \;=\; & aC_A\left[\frac{5}{8} - \frac{3}{8}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right] \\
& + a^2\left(\frac{\sqrt{C_A}T_FN_fm_q^2}{\gamma^2}\right)\left[4\omega + \frac{\pi}{2}\right] \\
& + a^2\left[C_A^2\left[\frac{2017}{768} - \frac{11097}{2048}s_2 + \frac{95}{256}\zeta(2) - \frac{65}{48}\overline{\ln}(C_A\gamma^4)\right.\right. \\
& \quad + \left.\frac{35}{128}\left(\overline{\ln}(C_A\gamma^4)\right)^2 + \frac{1137}{2560}\sqrt{5}\zeta(2) - \frac{205\pi^2}{512}\right] \\
& \quad + C_AT_FN_f\left[2\ln(2) - \frac{25}{24} + \frac{1}{2}\overline{\ln}^2(m_q^2) - \frac{1}{2}\overline{\ln}(m_q^2)\overline{\ln}(C_A\gamma^4)\right. \\
& \qquad + \frac{19}{12}\overline{\ln}(C_A\gamma^4) - \overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] \\
& \quad - \left.\left.\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] + \frac{\pi^2}{8}\right]\right] \\
& + a^2\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^2} + C_A\gamma^4}\left(\frac{\sqrt{C_A}T_FN_f}{\sqrt{2}\gamma^2}\right) \\
& \quad \times\left[-\frac{\zeta(2)}{4} - \frac{1}{2}\ln^2(2) - \frac{1}{2}\ln(2)\overline{\ln}(m_q^2) - \frac{1}{4}\ln(2)\ln(C_A\gamma^4)\right. \\
& \quad + \frac{1}{2}\ln(2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^2m_q^4} + C_A\gamma^4}\right] \\
& \quad + \frac{1}{2}\ln(2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^2m_q^4} + C_A\gamma^4}\right] - \frac{1}{8}\overline{\ln}^2(m_q^2) - \frac{1}{8}\overline{\ln}(m_q^2)\overline{\ln}(C_A\gamma^4) \\
& \quad + \frac{1}{4}\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] - \frac{1}{32}\overline{\ln}^2(C_A\gamma^4) \\
& \quad + \frac{1}{8}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] \\
& \quad - \frac{1}{8}\overline{\ln}^2\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] \\
& \quad + \frac{1}{4}\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] \\
& \quad + \frac{1}{8}\overline{\ln}(\sqrt{C_A}\gamma^2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] \\
& \quad - \frac{1}{4}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] \\
& \qquad \times\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] \\
& \quad - \left.\frac{1}{8}\overline{\ln}^2\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4m_q^4} + C_A\gamma^4}\right] + \frac{\pi}{4}\omega + \frac{\pi^2}{32}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ a^2 \left[\frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^2}+C_A\gamma^4}}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}}\right]\left(\frac{\sqrt{C_A}T_F N_f m_q^4}{\sqrt{2}\gamma^2}\right)\\
&\quad\times\Big[-\zeta(2)-2\ln^2(2)-2\ln(2)\overline{\ln}(m_q^2)-\ln(2)\overline{\ln}(C_A\gamma^4)\\
&\qquad+\ 2\ln(2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad+\ 2\ln(2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]-\frac{1}{2}\overline{\ln}^2(m_q^2)-\frac{1}{2}\overline{\ln}(m_q^2)\overline{\ln}(C_A\gamma^4)\\
&\qquad+\ \overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad-\ \frac{1}{8}\overline{\ln}^2(C_A\gamma^4)+\frac{1}{2}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad-\ \frac{1}{2}\overline{\ln}^2\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad+\ \overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad+\ \frac{1}{2}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad-\ \overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad\quad\times\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad-\ \frac{1}{2}\overline{\ln}^2\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]+2\omega^2-2\mathrm{Li}_2(r,\theta)+\pi\omega+\frac{\pi^2}{8}\Big]\\
&+ a^2 \left[\frac{\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^2}+C_A\gamma^4}}{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}}\right]\left(\frac{(C_A)^{3/2}T_F N_f \gamma^2}{\sqrt{2}}\right)\\
&\quad\times\Big[-\frac{\zeta(2)}{4}-\frac{1}{2}\ln^2(2)-\frac{1}{2}\ln(2)\overline{\ln}(m_q^2)-\frac{1}{4}\ln(2)\overline{\ln}(C_A\gamma^4)\\
&\qquad+\ \frac{1}{2}\ln(2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad+\ \frac{1}{2}\ln(2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]-\frac{1}{8}\overline{\ln}^2(m_q^2)-\frac{1}{8}\overline{\ln}(m_q^2)\overline{\ln}(C_A\gamma^4)\\
&\qquad+\ \frac{1}{4}\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]-\frac{1}{32}\overline{\ln}^2(C_A\gamma^4)\\
&\qquad+\ \frac{1}{8}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]\\
&\qquad-\ \frac{1}{8}\overline{\ln}^2\left[\sqrt{2}\sqrt{C_A}\gamma^2+\sqrt{\sqrt{C_A^2\gamma^8+16C_A\gamma^4 m_q^4}+C_A\gamma^4}\right]
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{1}{4}\overline{\ln}(m_q^2)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] \\
& + \frac{1}{8}\overline{\ln}(C_A\gamma^4)\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] \\
& - \frac{1}{4}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] \\
& \quad \times \overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] \\
& - \frac{1}{8}\overline{\ln}^2\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] + \frac{\omega^2}{2} - \frac{1}{2}\mathrm{Li}_2(r,\theta) + \frac{\pi\omega}{4} + \frac{\pi^2}{32}\Bigg] \\
+ \; & a^2\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^2} - C_A\gamma^4}\left(\frac{\sqrt{C_A}T_F N_f}{\sqrt{2}\gamma^2}\right) \\
\times & \Bigg[- \frac{\pi}{4}\ln(2) - \frac{\pi}{8}\overline{\ln}(m_q^2) - \frac{\pi}{16}\overline{\ln}(C_A\gamma^4) + \frac{\pi}{8}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] \\
& + \frac{\pi}{8}\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] \\
& + \frac{1}{4}\mathrm{Cl}_2(2\theta) - \frac{1}{4}\mathrm{Cl}_2(2\theta + 2\omega) + \frac{1}{4}\mathrm{Cl}_2(2\omega) \\
& + \left[\frac{m_q^4}{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4}}\right] \\
& \quad \times \Bigg[\pi\ln(2) + \frac{\pi}{2}\overline{\ln}(m_q^2) + \frac{\pi}{4}\overline{\ln}(C_A\gamma^4) - \frac{1}{2}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] \\
& \qquad - \frac{1}{2}\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] \\
& \qquad - \mathrm{Cl}_2(2\theta) + \mathrm{Cl}_2(2\theta + 2\omega) - \mathrm{Cl}_2(2\omega)\Bigg]\Bigg] \\
+ \; & a^2\left[\frac{\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} - C_A\gamma^4}}{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4}}\right]\left(\frac{(C_A)^{3/2}\gamma^2 T_F N_f}{\sqrt{2}}\right) \\
\times & \Bigg[\frac{1}{4}\mathrm{Cl}_2(2\theta + 2\omega) - \frac{1}{4}\mathrm{Cl}_2(2\theta) - \frac{1}{4}\mathrm{Cl}_2(2\omega) + \frac{\pi}{4}\ln(2) + \frac{\pi}{8}\overline{\ln}(m_q^2) \\
& + \frac{\pi}{16}\overline{\ln}(C_A\gamma^4) - \frac{\pi}{8}\overline{\ln}\left[\sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right] \\
& - \frac{1}{8}\overline{\ln}\left[\sqrt{2}\sqrt{C_A}\gamma^2 + \sqrt{\sqrt{C_A^2\gamma^8 + 16C_A\gamma^4 m_q^4} + C_A\gamma^4}\right]\Bigg] \; + \; O(a^3) \; .
\end{aligned}
\tag{4.1}
$$

This is a real expression and the main result of our article. There are several checks. Whilst we have been careful in checking that the four two loop scalar master integrals reduce to the correct expressions in the massless quark limit, the overall final gap equation must also satisfy the same test. We note that (4.1) does do this and for completeness note that one obtains

$$
1 \;=\; C_A\left[\frac{5}{8} - \frac{3}{8}\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right] a
$$

$$
\begin{aligned}
&+ \Bigg[C_A^2 \left(\frac{2017}{768} - \frac{11097}{2048} s_2 + \frac{95}{256}\zeta(2) - \frac{65}{48} \ln\left(\frac{C_A\gamma^4}{\mu^4}\right) + \frac{35}{128}\left(\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right)^2 \right. \\
&\qquad \left. + \frac{1137}{2560}\sqrt{5}\zeta(2) - \frac{205\pi^2}{512} \right) \\
&\quad + C_A T_F N_f \left(- \frac{25}{24} - \zeta(2) + \frac{7}{12} \ln\left(\frac{C_A\gamma^4}{\mu^4}\right) - \frac{1}{8}\left(\ln\left(\frac{C_A\gamma^4}{\mu^4}\right)\right)^2 + \frac{\pi^2}{8} \right) \Bigg] a^2 \\
&+ O(a^3) \qquad (4.2)
\end{aligned}
$$

where $s_2 = (2\sqrt{3}/9)\mathrm{Cl}_2(2\pi/3)$ which was originally recorded in [15]. However, there is another check on (4.1) which is to examine the Faddeev-Popov ghost 2-point function in the zero momentum limit. As was noted in [1] there ought to be ghost enhancement which equates to the Kugo-Ojima criterion being satisfied, [32]. Formally writing the radiative corrections to the Faddeev-Popov ghost 2-point function as $u(p^2)$ then ghost enhancement follows if $u(0) = -1$ which is the Kugo-Ojima condition. This was verified at two loops in the massless quark case in [16]. Therefore, we have repeated that calculation here with massive quarks and examined the zero momentum limit. This involves applying the vacuum bubble expansion to the 31 contributing two loop Feynman diagrams. The computation makes use of the master integrals discussed in section 3 and we have used the same routines in order to do the Form identifications. The outcome is similar to [16]. In other words the Kugo-Ojima criterion is satisfied at two loops precisely when the two loop massive quark Gribov gap equation is satisfied. Indeed as emphasised in Zwanziger's articles, the theory has no meaning as a gauge theory unless this occurs. Therefore, we are confident that our result (4.1) is correct.

5 Discussion.

We conclude with several observations. The inclusion of massive quarks in the Gribov-Zwanziger approach has not affected the main properties of the Faddeev-Popov ghost enhancement at two loops. Moreover, the one loop verification of gluon suppression of [16] is also unaffected with massive quarks. This is because at one loop the diagrams involving massive quarks do not arise in that part of the matrix of 2-point functions responsible for the vanishing of the gluon propagator in the infrared. It is worth noting that our original expectation was that massive quarks would not upset these key properties of the Yang-Mills fields. One of the main outcomes of the result (4.1) is the very much involved form which is clearly due to the multi-scale nature of the underlying Feynman diagram. Whilst we have concentrated on what is now known as the scaling solution, [17, 18, 19, 20, 21, 22], rather than the decoupling solution it does serve as an indication of what to expect if one were to study the same problem in the latter case. For instance, within the Gribov-Zwanziger context, [23, 24], the gluon propagator acquires an additional mass scale deriving from the condensation of a mass for the Zwanziger localizing ghost fields. Aside from giving three scale two loop integrals for Feynman graphs without quarks it will result in *four* scale two loop integrals for the case we studied in depth here. Clearly that would be a difficult computation.

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